

# Gravitating Yang-Mills vortices in 4+1 spacetime dimensions

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The coupling to gravity in D=5 spacetime dimensions is considered for the particle-like and vortex-type solutions obtained by uplifting the D=4 Yang-Mills instantons and D=3 Yang-Mills-Higgs monopoles. It turns out that the particles become completely destroyed by gravity, while the vortices admit a rich spectrum of gravitating generalizations. Such vortex defects may be interesting in view of the AdS/CFT correspondence or in the context of the brane world scenario.

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**1. Introduction.**— The recent interest in the AdS/CFT correspondence (see [1] for a review), and in the brane world scenario [10] has attracted attention to gravity theories in  $D=5$  spacetime dimensions. In this connection solutions of gauged supergravities have been actively studied. As such theories generically contain the non-Abelian gauge fields, considering solutions for such fields coupled to gravity seems to be important. At the same time, most studies so far have been restricted exclusively to the Abelian sector. Practically all what is known about gravitating non-Abelian solutions in  $D=5$  are the BPS configurations described in [6]. In  $D=4$  on the other hand, gravitating Yang-Mills (YM) fields have been analyzed in some detail (see [11] for a review), partially with the numerical methods.

The aim of this letter is to study more systematically gravity-coupled YM fields in  $D=5$  by applying the methods previously used in  $D=4$ . Instead of specializing to any particular supergravity model, we shall consider the pure Einstein-YM (EYM) theory. Although this theory is probably non-supersymmetric in itself, it enters all gauged supergravities as the basic building block, and hence one can expect some features of its solutions to be generic. In principle, one can study this theory also in higher dimensions, but we shall work only in  $D=5$ .

As is well known, the pure YM theory in  $D=5$  Minkowski space admits topologically stable, particle-like and vortex-type solutions obtained by uplifting the  $D=4$  YM instantons and  $D=3$  YM-Higgs monopoles. It is then natural to wonder what happens to these objects when they are coupled to gravity. This issue seems to be interesting in itself, and it has not been addressed before. Below we shall study this problem and find that the particles become completely destroyed by gravity, as a result of their scaling behavior, while the vortices admit very non-trivial gravitating generalizations. These gravitating vortices comprise an infinite family including the fundamental solution and its excitations. All solutions exist for a finite range of values of the gravitational coupling constant, and in the strong gravity limit they become gravitationally closed.

It seems plausible that such solutions could be further generalized to include scalars and a cosmological term. In this case they would describe stable vortex excitations over the  $\text{AdS}_5$  background. Such topological defects might be interesting in view of the AdS/CFT correspondence as providing the dual gravity description for some processes on the gauge theory side, or possibly in the context of the brane world scenario.

**2. The model.**— The pure Einstein-YM theory for the gauge group  $SU(2)$  is defined by the action

$$S = \int \left( -\frac{1}{16\pi G} R - \frac{1}{4g^2} F_{MN}^a F^{aMN} \right) \sqrt{g} d^5x. \quad (1)$$

Here  $F_{MN}^a = \partial_M A_N^a - \partial_N A_M^a + \varepsilon_{abc} A_M^b A_N^c$  ( $a = 1, 2, 3$ ), and  $[G^{1/3}] = [g^2] = [\text{length}]$ . Let us split the coordinates as  $x^M = (x^0, x^\mu)$ , where  $x^\mu = (x^i, x^4)$ . In the flat spacetime limit,  $G \rightarrow 0$ , the YM theory is not conformally invariant, the length scale being  $g^2$ , and this allows for soliton solutions. Specifically, for static, purely magnetic fields with  $A_M^a = (0, A_\mu^a(x^\nu))$  the energy  $E = \frac{1}{4g^2} \int (F_{\mu\nu}^a)^2 d^4x$  coincides with the action of the D=4 Euclidean YM theory. It follows then that there are regular, topologically stable solutions in D=5 with the energy  $E = 8\pi^2 |n|/g^2$ , where the topological winding number  $n \in \pi_3(SU(2))$ . These solutions describe neutral, particle-like objects which we shall call “YM instanton particles”. If  $\partial/\partial x^4$  is a symmetry, one can choose  $A_M^a = (0, A_i^a(x^k), H^a(x^k))$ , and the energy per unit  $x^4$ ,  $E = \frac{1}{2g^2} \int ((\partial_i H^a + \varepsilon_{abc} A_i^b H^c)^2 + \frac{1}{2} (F_{ik}^a)^2) d^3x$ , coincides with the energy of the D=3 YM-Higgs fields. Since  $H^a H^a$  is asymptotically constant [8], there are topological vortex-type solutions with the energy per unit length  $E = 4\pi |n|/g^2$ , where  $n \in \pi_2(S^2)$ . These we shall call “YM vortices”. Let us now set  $G \neq 0$ .

**3. Gravitating YM particles.**— We parameterize the  $SO(4)$ -invariant line element in the one-particle sector as

$$ds^2 = \sigma^2 N dt^2 - \frac{dr^2}{N} - r^2 d\Omega_3^2, \quad (2)$$

with  $N \equiv 1 - \kappa m/r^2$ , the length scale being  $g^2$ . The gauge field is given by  $A^a = (1 + w)\theta^a$ . Here  $w$ ,  $\sigma$ ,  $m$  are functions of  $r$ ,  $d\Omega_3^2 = \theta^a \theta^a$  is the line element of the unit sphere  $S^3$ , and  $\theta^a$  are the invariant forms on  $S^3$ , such that  $d\theta^a + \varepsilon_{abc} \theta^b \wedge \theta^c = 0$ . The gravitational coupling constant is  $\kappa = 8\pi G/g^6$ . The ADM mass,  $M = m(\infty)$ , determines the dimensionful energy  $E_{\text{ADM}} = (3\pi^2/g^2)M$ . The independent field equations read

$$\begin{aligned} r^2 N w'' + r w' + \kappa (m - (w^2 - 1)^2) \frac{w'}{r} &= 2(w^2 - 1)w, \\ r m' &= r^2 N w'^2 + (w^2 - 1)^2, \end{aligned} \quad (3)$$

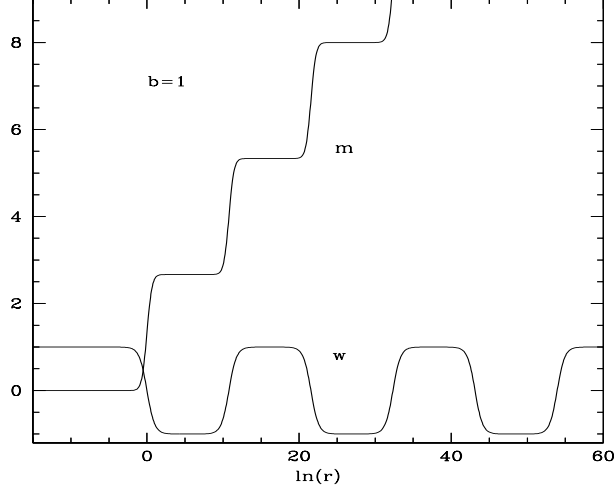


Figure 1: Typical solution to Eqs.(3) (with  $\kappa = 10^{-8}$ )

while the equation for  $\sigma$  decouples and can be integrated giving  $\sigma(r) = \exp\left(-\kappa \int_r^\infty \frac{w'^2}{r} dr\right)$ . It follows that  $r(m\sigma)' = (r^2 w'^2 + (w^2 - 1)^2)\sigma$ , which can be integrated with the boundary conditions  $m(0) = 0$  to give

$$M[w, \kappa] = \int_0^\infty \frac{dr}{r} (r^2 w'^2 + (w^2 - 1)^2) \exp\left(-\kappa \int_r^\infty \frac{w'^2}{r} dr\right). \quad (4)$$

This is the ADM mass for configurations subject to the (00) and (rr) Einstein equations, and for such fields it is proportional to the action density. It follows that on-shell fields are stationary points of  $M[w, \kappa]$ . For  $\kappa = 0$  the integrand in (4) can be represented as a sum of a total derivative and a perfect square. The latter vanishes if the self-duality equation holds,  $rw' = \pm(w^2 - 1)$ , whose solutions are  $w = \mp \frac{br^2 - 1}{br^2 + 1}$ , where  $b$  is a scale parameter. These describe the regular “instanton particles” with the energy  $E_{\text{ADM}} = 8\pi^2/g^2$ .

It turns out that for  $\kappa \neq 0$  Eqs.(3) do not admit globally regular solutions with finite ADM mass. Indeed, for such solutions  $\sigma(r) \neq 0$ , such that the exponent in (4) never vanishes, and therefore (4) can only be finite if  $w(r) \rightarrow \pm 1$  for  $r \rightarrow 0, \infty$ . It follows then that  $M[w(\lambda r), \kappa] = M[w(r), \lambda^2 \kappa]$ , where  $\lambda$  is a constant scaling parameter. Since  $M$  should be stationary with respect to rescalings, one should have  $0 = \frac{d}{d\lambda} M[w(r), \lambda^2 \kappa]|_{\lambda=1}$ , but the latter is manifestly negative for  $\kappa \neq 0$ .

The YM particles therefore do not generalize to curved space – as a consequence of their scaling behavior. Specifically, since the size of the particles can be arbitrary, there is the corresponding scaling zero mode of the energy functional. This does not allow one to apply the inverse function theorem [9] to construct the gravitating generalizations even for arbitrarily small  $\kappa$ . Qualitatively, the YM particles resemble dust, since their rescaling changes the gravitational binding (the exponential factor in (4)), without affecting the energy of the gauge field (the rest of the integrand in (4)). As a result, the attraction and repulsion are not balanced and the equilibrium is impossible – the gravitating system wants to shrink but there is no pressure to stop the contraction. Notice that these arguments apply only in D=5. In D=4 the YM instantons can be coupled to gravity without problems – since their energy-momentum tensor then vanishes, they fulfill the coupled system of the Euclidean EYM equations for any value of  $G$ .

One can numerically integrate Eqs.(3) with the regular boundary conditions at the origin,  $w = 1 - 2br^2 + O(r^4)$ ,  $m = O(r^3)$ , in order to see what actually happens to the static YM particle solution if  $\kappa \neq 0$ . It turns out that as  $r$  increases,  $w$  first follows very closely (for small  $\kappa$ ) the flat space configuration, but then it fails to reach the value  $-1$  as  $r \rightarrow \infty$  and starts replicating the same pattern infinitely many times. As a result there emerges the quasi-periodic structure shown in Fig.1. This can be explained: integrating the Yang-Mills equation in (3) gives

$$w'^2 - (w^2 - 1)^2 = - \int_{-\infty}^{\tau} (\ln(\sigma^2 N))' w'^2 d\tau, \quad (5)$$

with  $' \equiv \frac{d}{d\tau} \equiv r\sqrt{N} \frac{d}{dr}$ . This describes a particle moving with friction in the inverted double-well potential  $U = -(w^2 - 1)^2$ . At  $\tau = -\infty$  the particle starts at the local maximum of the potential at  $w = 1$ , but then it loses energy due to the dissipation and cannot reach the second maximum at  $w = -1$ . Hence it bounces back. For large  $\tau$  the dissipative term tends to zero, and the particle ends up oscillating in the potential well with a constant energy. After each oscillation the mass function  $m$  increases in a step-like fashion, and for large  $r$  one has  $m \sim \tau \sim \ln r$ . As a result, there emerges an infinite sequence of static spherical shells of the YM energy in the D=5 spacetime.

**4. Gravitating YM vortices.**– We parameterize the SO(3)-invariant metric in the one-vortex sector as

$$ds^2 = e^{2\nu} dt^2 - e^{2\lambda} dr^2 - e^{2\mu} d\Omega_2^2 - e^{2\zeta} (dx^4)^2, \quad (6)$$

with  $d\Omega_2^2 = d\vartheta^2 + \sin^2 \vartheta d\varphi^2$ . The  $\text{SO}(3)$ -invariant gauge field is given by

$$A^a = (w - 1) \varepsilon_{abc} n^b dn^c + H n^a dx^4, \quad (7)$$

where  $n^a = (\sin \vartheta \cos \varphi, \sin \vartheta \sin \varphi, \cos \vartheta)$ . Here  $\nu, \lambda, \mu, \zeta, w$ , and  $H$  are functions of  $r$ . With  $s = e^{\nu+\zeta+2\mu-\lambda}$  the field equations read

$$s (s e^{-2\mu} w')' - e^{2\nu+2\zeta} (w^2 - 1) w = e^{2\nu+2\mu} H^2 w, \quad (8)$$

$$s (s e^{-2\zeta} H')' = 2e^{2\nu+2\mu} w^2 H, \quad (9)$$

$$s (s \nu')' = \frac{\kappa}{3} (A + 2B + 2C + D), \quad (10)$$

$$s (s \zeta')' = \frac{\kappa}{3} (-2A + 2B - 4C + D), \quad (11)$$

$$s (s \mu')' - e^{2\nu+2\zeta+2\mu} = \frac{\kappa}{3} (A - B - C - 2D), \quad (12)$$

$$\begin{aligned} s^2 (\mu'^2 + 2\nu' \mu' + 2\zeta' \mu' + \zeta' \nu') - e^{2\nu+2\zeta+2\mu} \\ = \frac{\kappa}{2} (A + 2B - 2C - D). \end{aligned} \quad (13)$$

Here  $A = e^{-2\zeta} s^2 H'^2$ ,  $B = e^{-2\mu} s^2 w'^2$ ,  $C = e^{2\nu+2\mu} w^2 H^2$ ,  $D = e^{2\nu+2\zeta} (w^2 - 1)^2$  and  $' \equiv \frac{d}{dr}$ . In this system the last equation is the initial value ‘‘Gauss’’ constraint generating the residual gauge symmetry  $r \rightarrow \tilde{r}(r)$ . This symmetry can be used to impose a gauge condition on the amplitudes. The equations are also invariant with respect to the rescalings of the  $t$  and  $x^4$  coordinates,  $\nu \rightarrow \nu + \nu_0$ ,  $\zeta \rightarrow \zeta + \zeta_0$ ,  $H \rightarrow e^{\zeta_0} H$ , with constant  $\nu_0$  and  $\zeta_0$ .

In addition, the equations are invariant with respect to the dilatations,  $\mu \rightarrow \mu + \epsilon$ ,  $\lambda \rightarrow \lambda + \epsilon$ ,  $\zeta \rightarrow \zeta + \epsilon$ ,  $\kappa \rightarrow e^{2\epsilon} \kappa$ , with constant  $\epsilon$ . The associated conserved Noether charge is

$$\mathcal{Q} = s(-\nu' + \zeta' + \kappa H H' e^{-2\zeta}). \quad (14)$$

The origin of this symmetry can be traced to the fact that effectively the system can be viewed as the D=4 EYM-Higgs theory coupled to a dilaton. In general, if nothing depends on  $x^4$ , one can parameterize the 5-fields as  $\mathbf{g}_{MN} dx^M dx^N = e^{-\zeta} \gamma_{\mu\nu} dx^\mu dx^\nu - e^{2\zeta} (dx^4)^2$  and  $A_M^a dx^M = A_\mu^a dx^\mu + H^a dx^4$ . The action (1) then reduces to  $S = \frac{1}{g^2} \int dx^4 \int d^4 x \sqrt{-\gamma} \mathcal{L}_4$  with

$$\mathcal{L}_4 = -\frac{1}{2\kappa g^4} \mathcal{R} + \frac{3}{\kappa g^4} \partial_\mu \zeta \partial^\mu \zeta - \frac{1}{4} e^\zeta F_{\mu\nu}^a F^{a\mu\nu} + \frac{1}{2} e^{-2\zeta} \mathcal{D}_\mu H^a \mathcal{D}^\mu H^a. \quad (15)$$

Here  $\mathcal{R}$  is the Ricci scalar for  $\gamma_{\mu\nu}$ , the indices are lifted by  $\gamma^{\mu\nu}$ , also  $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + \varepsilon_{abc} A_\mu^b A_\nu^c$  and  $\mathcal{D}_\mu H^a = \partial_\mu H^a + \varepsilon_{abc} A_\mu^b H^c$ . This determines the theory of coupled EYM-Higgs-dilaton fields in D=4. The amplitude  $\zeta$  therefore effectively plays the role of the dilaton, which explains the dilatational symmetry. The vortex solutions under consideration thus can be viewed as D=4 gravitating YM-Higgs monopoles coupled to the dilaton in a special way (the systems considered so far in the literature do not have the direct coupling between the dilaton and Higgs fields [5].)

Some simplest solutions of the system (8)–(13) can be found. The vacuum Schwarzschild solution is obtained for  $w = \pm 1$ ,  $H = 0$ ,  $e^{-2\lambda} = 1 - \frac{4\kappa M}{3r}$ , and either  $\nu = -\lambda$ ,  $\zeta = 0$  or  $\zeta = -\lambda$ ,  $\nu = 0$ . The extreme charged Abelian black string solution is described by  $w = 0$ ,  $H = H_0$ ,  $e^{2\nu} = e^{2\zeta} = e^{-\lambda} = 1 - r_H/r$ , with  $r_H = \sqrt{2\kappa/3}$ .

We are interested in globally regular solutions with finite mass per unit  $x^4$ . This implies that  $\nu = \lambda = \zeta = 0$  at  $r = \infty$ , and if  $e^\mu = r$  then  $\nu = -\frac{2\kappa}{3r}M + O(r^{-2})$  for large  $r$ , while  $\lambda = \frac{\kappa}{3r}M_{\text{ADM}} + O(r^{-2})$ . Here  $M$  and  $M_{\text{ADM}}$ , respectively, are the Newtonian and the ADM masses, the dimensionful energy being  $E = (4\pi/g^2)M$ . Choosing  $e^\mu = r$ , dividing Eq.(10) by  $(\kappa s)$  and integrating gives

$$M = \int_0^\infty dr e^\nu \left( \frac{r^2}{2} e^{-\zeta-\lambda} H'^2 + e^{-\lambda} w'^2 + e^{\lambda-\zeta} w^2 H^2 + e^{\lambda+\zeta} \frac{(w^2 - 1)^2}{2r^2} \right). \quad (16)$$

In flat space, with  $\nu = \lambda = \zeta = 0$ , the integrand here can be rearranged as the sum of a total derivative and two perfect squares. The latter vanish if the Bogomol'nyi equations hold,  $r^2 H' + w^2 - 1 = 0$  and  $w' + wH = 0$ . The BPS solution,  $H = \coth r - 1/r$  and  $w = r/\sinh r$ , describes the flat space vortex with  $M = M_{\text{ADM}} = 1$ .

The Noether charge  $\mathcal{Q}$  vanishes for globally regular solutions. As a result, using (14) with  $\mathcal{Q} = 0$  allows us to exclude  $\nu$  from the equations. Next, introducing  $h \equiv e^{-\zeta} H$  the amplitude  $\zeta$  can also be eliminated. Choosing the radial gauge where  $\lambda = 0$  and defining  $R \equiv e^\mu$ , the independent variables are  $w$ ,  $h$ ,  $Z \equiv \zeta'$ , and  $R$ . If these are determined, the amplitudes  $\nu$  and  $\zeta$  are given by  $\zeta = \int Z dr$ , and  $\nu = \zeta + \kappa \int (Zh^2 + hh') dr$ . The field equations (8)–(12) can be reformulated as a seven-dimensional autonomous system

$$\frac{d}{dr} y_k = F_k(y_s, \kappa), \quad (17)$$

with  $y_k = \{w, w', h, h', Z, R, R'\}$ , where the functions  $F_k(y_s, \kappa)$  can be read-off from (8)–(12). In addition, the constraint (13) will restrict the initial values of the  $y_k$ 's. Eqs.(17) possess the dilatational symmetry

$$r \rightarrow \epsilon r, \quad w \rightarrow w, \quad h \rightarrow h/\epsilon, \quad Z \rightarrow Z/\epsilon, \quad R \rightarrow \epsilon R, \quad \kappa \rightarrow \epsilon^2 \kappa, \quad (18)$$

which also changes the mass as  $M \rightarrow M/\epsilon$ . The following fixed points of the equations will be important in our analysis:

I. The origin,  $(w, h, Z, R) = (1, 0, 0, 0)$ . Here the constraint “stable manifold” of solutions that are regular for  $r \rightarrow 0$  can be characterized by the Taylor expansions

$$\begin{aligned} w &= 1 - br^2 + O(r^2), & h &= ar + O(r^3), \\ Z &= O(r^2), & R &= r + O(r^3). \end{aligned} \quad (19)$$

II. Infinity,  $(w, h, Z, R) = (0, v, 0, \infty)$ , where the (unconstrained) stable manifold can be described by

$$\begin{aligned} w &= A r^{Cv} e^{-vr} + o(e^{-r}), & Z &= \kappa Q r^{-2} + O(r^{-3} \ln r), \\ h &= v(1 - Cr^{-1}) + O(r^{-2} \ln r), \\ R &= r - m \ln r + m^2 r^{-1} \ln r - r_\infty + \gamma r^{-1} + O(r^{-2} \ln r). \end{aligned} \quad (20)$$

Here  $m \equiv \kappa(Cv^2 + (2 + \kappa v^2)Q)$  determines the ADM mass,  $M_{\text{ADM}} = \frac{3}{\kappa} m$ , while  $M = \frac{1}{2} M_{\text{ADM}} - \frac{3}{2} Q$ . In (19),(20)  $a, b, A, v, C, Q, r_\infty, \gamma$  are eight free parameters, but due to the scaling symmetry (18) only seven are independent, and we shall set  $v = 1$ .

III. “Warped”  $AdS_3 \times S^2$ . This fixed point is determined by the real root of the algebraic equation  $4q^3 + 7q^2 + 11q = 1$ :

$$w^2 = q, \quad R^2 = \kappa \frac{(11q - 1)(1 - q)}{(4q^2 - 13q + 1)}, \quad h^2 = \frac{(1 - q)}{R^2}, \quad Z^2 = -\frac{(4q^2 - 13q + 1)}{(4q + 1)R^2}. \quad (21)$$

Evaluating,

$$w = 0.29, \quad h = \frac{1.27}{\sqrt{\kappa}}, \quad Z = \pm \frac{0.31}{\sqrt{\kappa}}, \quad R = 0.75\sqrt{\kappa}. \quad (22)$$

This is a new exact, essentially non-Abelian solution to the field equations. Its geometry is

$$ds^2 = e^{2\alpha Z r} dt^2 - dr^2 - e^{2Z r} (dx^4)^2 - R^2 d\Omega_2^2, \quad (23)$$



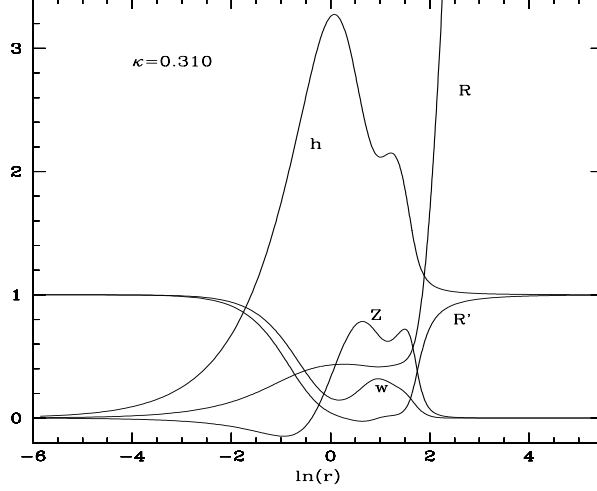


Figure 2: The vortex solution in the strong gravity regime.

with  $\alpha = 1 + \kappa h^2 = 2.62$  (notice that for  $\alpha = 1$  this would be the metric on  $AdS_3 \times S^2$ ). The gauge field is given by (7) with  $H = e^{Zr}h$ . Linearizing Eqs.(17) around this solution, one finds the 7 characteristic eigenvalues of the linearized system to be  $\{-2.77, -2.47, -2.12, -0.61 \pm 1.24i, 0.88, 1.54\}$ , in units of  $1/\sqrt{\kappa}$ . The solution (21) is therefore a hyperbolic (saddle) fixed point with five stable and two unstable for  $r \rightarrow \infty$  eigenmodes.

We now integrate the equations (17) for a given  $\kappa \neq 0$  in the interval  $r \in [0, \infty)$  using (19),(20) as the boundary conditions and adjusting the seven independent free parameters to match the seven  $y_k$ 's. This gives the gravitating vortex solutions. For small  $\kappa$  they are only slightly deformed as compared to the flat space vortex. The essentially non-Abelian field is contained in the central core where  $w \neq 0$ , while in the far field zone  $w$  vanishes and only the long range U(1) component of the gauge field survives. The metric amplitudes are  $Z \approx 0$ ,  $R \approx r$ ,  $R' \approx 1$ .

It turns out to be convenient to measure the coupling to gravity not by  $\kappa$  but by  $\xi = a\kappa$  where  $a$  comes from (19). Solutions are specified by their asymptotic parameters in (19),(20), which lie on the curve  $\Gamma(\xi)$  in the eight-dimensional parameter space with coordinates  $\Gamma = (\kappa, a, b, A, C, Q, r_\infty, \gamma)$ . As  $\xi$  increases and  $\Gamma(\xi)$  deviates from  $\Gamma(0)$ , the solutions deviate more and more from their flat space profiles:  $R'(r)$  develops more and more pronounced

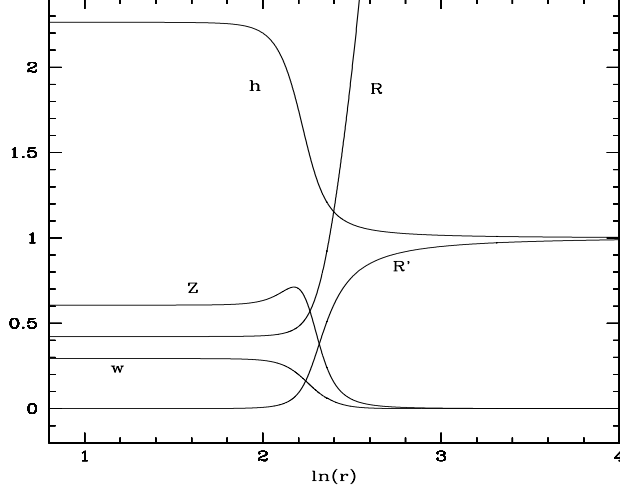


Figure 3: The exterior limiting solution.

minimum at some finite  $r$  where  $Z(r)$  considerably deviates from zero; see Fig.2. It is interesting that  $\kappa$  not always increases with  $\xi$ , but only up to the maximal value  $\kappa_{\max} = 3.22$  and then starts decreasing; see Fig.4 where projections of  $\Gamma(\xi)$  on the  $C\kappa$ ,  $CQ$ ,  $CA$ , and  $C\gamma$  planes are shown.

For strongly gravitating solutions, as  $\xi$  continues to increase, the amplitudes  $w(r)$ ,  $h(r)$ ,  $Z(r)$ ,  $R(r)$  start developing oscillations around the constant values which are close to those given by Eq.(22). The number of oscillations grows as gravity gets stronger, while their amplitudes tend to zero. The proper length of the interval where oscillations take place increases, and solutions develop a long throat connecting the core of the vortex with the asymptotic region. In the strong gravity limit this throat gets infinitely long, and the vortex core becomes disconnected from the outside world. A similar phenomenon has been observed in D=4 [3, 4].

In order to qualitatively understand such a behavior, each solution can be viewed as a trajectory interpolating in the phase space between the two fixed points (19) and (20). On its way, it gets attracted by the saddle point (21), due to the five stable for  $r \rightarrow \infty$  eigenmodes around this point, it spends some “time” in its vicinity, but finally it gets repelled due to the two unstable eigenmodes. As gravity gets stronger with growing  $\xi$ , the trajectory approaches closer and closer the saddle point (21) oscillating longer and

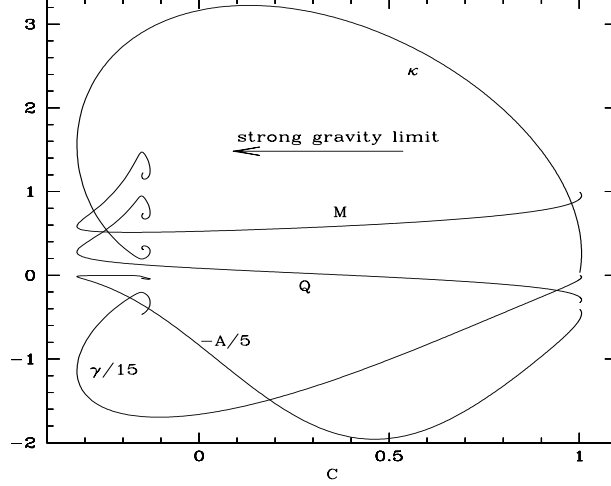


Figure 4: Parameters of the gravitating vortex solutions. The right ends of the curves correspond to the flat space vortex,  $\xi = 0$ . As  $\xi$  increases, the curves extend to the left, and for large  $\xi$  they spiral towards the values corresponding to the limiting solution.

longer in its vicinity. Finally, the trajectory exactly hits the saddle point and splits into two parts. The first part starts at the origin (19) and after infinitely many oscillations arrives at the saddle point (21). The second part interpolates between the saddle point and infinity (20). As a result, the vortex solution splits in the limit into two independent solutions. The first, interior solution contains the regular core, but asymptotically it is not flat and approaches instead the geometry (23). The second, exterior solution interpolates between (23) and the asymptotically flat region.

One can directly construct the exterior limiting solution. This lives in the interval  $r \in (-\infty, \infty)$ . Shifting  $r$  to set  $r_\infty = 0$  leaves four free coefficients in (20), while the (constraint) set of solutions that approach (22) for  $r \rightarrow -\infty$  is determined by the two unstable modes around this fixed point. There are altogether six free parameters, and the matching conditions can be fulfilled if only  $\kappa$  is treated as the seventh free parameter. The solution is found for  $\kappa = 0.316$ ; see Fig.3. This solution can be viewed as an extreme non-Abelian black string: in  $r = R$  coordinates  $e^{-2\lambda}$  has a double zero at  $r_h = 0.42$ , while  $e^{2\nu} \sim (r - r_h)^{2.02}$  and  $e^{2\zeta} \sim (r - r_h)^{0.77}$  for  $r \rightarrow r_h$ .

It is instructive to consider the behavior of the parameters of the solutions  $(\kappa, a, b, A, C, Q, r_\infty, \gamma)$  with increasing  $\xi$ . One can shift  $r \rightarrow r - r_\infty$  to set  $r_\infty = 0$ , which changes  $A \rightarrow Ae^{r_\infty}$ ,  $\gamma \rightarrow \gamma + mr_\infty$  without affecting the other parameters. One can then plot all parameters against  $\xi$ , but it is more illustrative to pick up one parameter,  $C$ , say, and plot the remaining ones against it. The resulting curves (some of them are shown in Fig.4) exhibit the characteristic spiraling behavior in the strong gravity limit (large  $\xi$ ). A similar oscillatory behavior of parameters in the critical limit is known for relativistic/boson stars [7].

Apart from the fundamental solutions, one finds also their excitations, for which  $w$  oscillates around zero. These can be parameterized by the number of nodes  $n = 1, 2, \dots$  of  $w$ . These solutions do not have the flat space limit: for  $\kappa \rightarrow 0$  their mass  $M \sim 1/\sqrt{\kappa}$ . This limit can be analyzed with the rescaling (18) with  $\epsilon = 1/\sqrt{\kappa}$ , which eliminates the explicit  $\kappa$ -dependence from the equations. The rescaled Higgs field then decouples in the limit, and this leads to solutions which resemble the D=4 Bartnik-McKinnon solutions [2] – with  $w$  oscillating around zero and  $w(\infty) = \pm 1$ .

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